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ABSTRACT. Discussion of the field of a disturbed flow produced by the unsteady motion of a thin foil or of the surface of an infinite circular cylindrical shell situated in a uniform equilibrium gas flow. Relevant equations are derived and analyzed.

1. We shall survey the field of disturbed flow caused by the unsteady /37* motion of a thin airfoil located in a uniform and equilibrium stream of gas moving with velocity U_∞ along the positive x axis in the system of stationary coordinates x, z. Let us suppose that in this field of turbulent flow there takes place a nonequilibrium process such as, for example, relaxation of the internal degree of freedom of a molecule or the reaction of dissociation of a diatomic gas. We shall neglect the effects of viscosity, heat conductivity and diffusivity. On account of the thinness of the airfoil and of the small deviation from some intermediate position, the disturbances introduced into the stream by the unsteady motion of the airfoil will be minimal, and the flow will differ little from the state of thermodynamic equilibrium. As is known [1, 2], a disturbed flow in this approximation is irrotational and the problem is reduced to the integration of the equation

$$T_\infty\left(\frac{\partial}{\partial t}+U_\infty\frac{\partial}{\partial x}\right)\left(\lambda_f^2\frac{\partial^2\Phi}{\partial x^2}-\frac{\partial^2\Phi}{\partial z^2}+2\frac{U_\infty}{a_{f\infty}^2}\frac{\partial^2\Phi}{\partial x\partial t}+\frac{1}{a_{f\infty}^2}\frac{\partial^2\Phi}{\partial t^2}\right)+$$
$$+\left(\lambda_e^2\frac{\partial^2\Phi}{\partial x^2}-\frac{\partial^2\Phi}{\partial z^2}+2\frac{U_\infty}{a_{e\infty}^2}\frac{\partial^2\Phi}{\partial x\partial t}+\frac{1}{a_{e\infty}^2}\frac{\partial^2\Phi}{\partial t^2}\right)=0$$
$$(\lambda_f^2=M_f^2-1,\ \lambda_e^2=M_e^2-1) \qquad (1.1)$$

Here: T_∞ = a parameter which is proportional to the relaxation time Θ_∞ (the subscript ∞ is used to designate quantities that correspond to the undisturbed stream); Φ = turbulent velocity potential; $M_f = U_\infty/a_{f\infty}$; $M_e = U_\infty/a_{e\infty}$; and $a_{f\infty}$ and $a_{e\infty}$ = the stagnation and equilibrium velocities of sound, respectively.

Since a_f is always greater than a_e, $M_f < M_e$. The boundary condition which dictates the requirement that the stream must move tangentially around the airfoil contours has the form

$$\frac{\partial\Phi}{\partial z}=\frac{\partial Z}{\partial t}+U_\infty\frac{\partial Z}{\partial x} \qquad \text{when } z = 0 \qquad (1.2)$$

where Z = Z(x,t) is the motion equation of the airfoil. Besides this, the po-

*Numbers in the margin indicate pagination in the foreign text.

tential Φ must satisfy the condition that the disturbances are attentuated at infinity. The expression for the pressure has the form

$$p = -\rho_\infty \left(\frac{\partial \Phi}{\partial t} + U_\infty \frac{\partial \Phi}{\partial x}\right) \tag{1.3}$$

where ρ_∞ = the **density of the gas in the undisturbed stream.**

As a simple example of the effect of nonequilibrium on the characteristics of the flow, let us examine the case where an infinite surface located at z = 0 is deformed according to the travelling wave law

$$Z = Z_0 \exp[ik(ct-x)] \qquad (k = 2\pi/\lambda) \tag{1.4}$$

Here: k = wave number; λ = wavelength; c = velocity of wave propagation. /38

If we search for the potential Φ in the form

$$\Phi = \Phi_0(z)\exp[tk(ct-x)]$$

then, solving the boundary problem (1.1)-(1.3), while taking into consideration (1.4) for the **potential Φ and the pressure,** we have

[illegible]

The dimensionless parameter $\Gamma = T_\infty U_\infty k$ represents the relationship between the characteristic relaxation time and the oscillatory motion. The unsteady flow is in equilibrium as $\Gamma \to 0$ and is constant as $\Gamma \to \infty$. For the pressure gradient at the **surface, we** have

$$p(x, -0) - p(x, +0) = [illegible] \tag{1.5}$$

2. For a description of a disturbed nonequilibrium field of flow caused by the unsteady motion of the surface of an infinite circular cylindrical shell of radius R, located in an evenly distributed and equilibrium stream of gas moving along the x axis of the shell with the velocity U_∞^i inside the shell and U_∞^e in the outer region (the parameters of flow inside the shell and in the outer region will be designated by the superscripts i and e, respectively), we

will use the equations for the turbulent velocity potential in the cylindrical coordinate system x, θ, and r (the x axis being directed along the axis of the shell)

$$T_\infty\left(\frac{\partial\Delta_j}{\partial t} + U_\infty \frac{\partial [illegible]}{\partial x}\right) [illegible] - \frac{1}{r}\frac{\partial\Phi}{\partial r} -$$
$$-\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + 2\frac{U_\infty}{a_{j\infty}^2}\frac{\partial^2\Phi}{\partial x\partial t} + \frac{1}{a_{j\infty}^2}\frac{\partial^2\Phi}{\partial t^2}, \quad j = f \text{ or } e \tag{2.1}$$

The boundary conditions on the moving wall shall be

$$\frac{\partial\Phi^i}{\partial r} = \frac{\partial w}{\partial t} + U_\infty^i\frac{\partial w}{\partial x} \text{ when } r = R - 0, \quad \frac{\partial\Phi^e}{\partial r} = \frac{\partial w}{\partial t} + U_\infty^e\frac{\partial w}{\partial x} \text{ when } r = R + 0$$

where $r = w(x,\theta,t)$ is the motion equation of the surface of the shell. At infinity, the potential Φ^e must satisfy the condition that the disturbances are attenuated, $\Phi^e \to 0$, and $\partial\Phi^e/\partial r \to 0$, as $r \to \infty$. If the potential Φ^e describes a wave process, these conditions must be replaced by the radiation conditions $\Phi^e = O(r^{-\frac{1}{2}})$, i.e., the wave intensity must decrease by the relation $r^{-\frac{1}{2}}$ as the distance from the source lengthens. Besides this, the potential should describe divergent waves if the shell radiates energy, and convergent waves if the shell absorbs energy from the stream [3].

Let us suppose that the surface of the shell is deformed according to the law $w = w_0\exp[ik(ct-x)]\cos n\theta$. Then, by representing the potential in the form of $\Phi = \Phi_1(r)\exp[ik(ct-x)]\cos n\theta$, we obtain the following from equation (2.1):

$$\frac{d^2\Phi_1}{dr^2} + \frac{1}{r}[illegible] \tag{2.2}$$

The solution of equation (2.2) appears in the form of the following Bessel functions:

$$\Phi_1 = C_1 J_n(\nu r) + C_2 N_n(\nu r) \quad \text{or} \quad [illegible] + C_2 H_n^{(2)}(\nu r)$$

Thus, the solution of the given problem is derived from the corresponding solution for an inert gas [3] by interchanging the value of $\nu^2 = k^2(M_e^2\zeta^2-1)$ with the value of (2.2).

3. Let us introduce the dimensionless coordinates x and z, referred to the quantity $l_r = T_\infty U_\infty$, which is proportional to the length of the relaxation, and the dimensionless time t, referred to T_∞. Let us suppose that the thin air- /39

foil accomplishes small harmonic vibrations of frequency ω in a supersonic evenly distributed and equilibrium stream of gas moving with the velocity U_∞ along the horizontal x axis. Let us represent the motion equations of the airfoil and the turbulent velocity potential in the form

$$Z = l_r h_a(x)\exp(i\Gamma t), \qquad \Phi = U_\infty l_r \varphi(x, z)\exp(i\Gamma t) \tag{3.1}$$

where $h_a(x)$ and (x,z) are the antisymmetric dimensionless components in the expansions of Z and Φ into their symmetric and antisymmetric parts with respect to the surface $z = 0$ (i.e., we are considering an antisymmetric problem here; see, for example [4]). The dimensionless parameter $\Gamma + \omega T_\infty$ represents, as it did above, the characteristic relaxation times and oscillatory motion, and determines the degree of nonequilibrium of the unsteady flow. Substituting (3.1) into (1.1)-(1.3), we obtain the following equations in the dimensionless coordinates x and z, which describe the disturbed harmonic nonequilibrium flow field:

$$\frac{\partial \Delta_{ij}}{\partial x} + i\Gamma\Delta_{ij} + \Delta_{ss} \text{ [illegible] } + 2i\Gamma M_f^2\frac{\partial\varphi}{\partial x} - \Gamma^2 M_f^2\varphi \tag{3.2}$$

$$\frac{\partial\varphi}{\partial z} = i\Gamma h_a \text{ [illegible] } \quad \text{when } z = 0, \quad \text{[illegible]} = -2\left(i\Gamma\varphi + \frac{\partial\varphi}{\partial x}\right) \tag{3.3}$$

We shall subsequently use new independent variables for the solution

[illegible]

With the new variables, the expressions (3.2) and (3.3) take the form

$$\left(i\Gamma + \frac{\partial}{\partial\xi}\right)\Delta_{ij} + (a - 1)\frac{\partial^2 f}{\partial\xi^2} + \Delta_{ss} = 0, \qquad \Delta_{ij} = 2\frac{\partial^2 f}{\partial\eta\,\partial\xi} - \frac{\partial^2 f}{\partial\eta^2} + \text{[illegible]} \tag{3.4}$$

$$\lambda_f\left(\text{[illegible]}\right) \quad \text{when } \eta = 0, \quad \text{[illegible]} \tag{3.5}$$

On the strength of the supersonic character of the flow, the disturbances will be equal to zero when $x < 0$. Then, using the Laplace transform

$$L[f(\xi, \eta)] = F(s, \eta) = \int_0^\infty f(\xi, \eta)\exp(-s\xi)\,d\xi$$

to equation (3.4), we obtain

$$(1 + i\Gamma + s[illegible] 2i\Gamma \frac{M_f^2}{\lambda_f^2} s(i\Gamma + s) -$$
$$- \Gamma^2 \frac{M_f^2}{\lambda_f^2}(i\Gamma + s) + 2i\Gamma \frac{M_e^2}{\lambda_f^2} s - \Gamma^2 \frac{M_e^2}{\lambda_f^2}\Big] F = -\Big[2\frac{\partial}{\partial\eta} + 2i\Gamma\frac{M_f^2}{\lambda_f^2} +$$
$$+ (s-1)\Big]\frac{\partial f(0,\eta)}{\partial\xi} + \Big[\frac{\partial^2}{\partial\eta^2} - 2(1 + i\Gamma + s)\frac{\partial}{\partial\eta} - 2i\Gamma\frac{M_f^2}{\lambda_f^2}(i\Gamma + s) -$$
$$- (s-1)s + \Gamma^2\frac{M_f^2}{\lambda_f^2} - 2i\Gamma\frac{M_e^2}{\lambda_f^2}\Big] f(0,\eta)$$

Equating the right side of this equation to zero (the validity of this as- /40
sumption will be proved below), we have

$$\frac{d^2F}{d\eta^2} - [illegible] - \Gamma^2\frac{M_f^2}{\lambda_f^2} \quad [illegible] \qquad (3.6)$$

The solution of (3.6), which satisfies the transformed boundary condition

$$\lambda_f\Big(\frac{dF}{d\eta} - [illegible] \omega_a(s) \quad [illegible] \omega_a(\xi))\Big)$$

and the condition of finiteness of disturbances at infinity, will be

$$[illegible] M_f^2)\Big]^{1/2} \qquad (3.7)$$

Let us investigate the disturbed field flow near the first stagnation Mach line, i.e., when $\xi \to 0$. To find the values of the potential f and the velocity, $u = \phi_x = f_\xi$ and $v = \phi_z = \lambda_f(f_\eta - f_\xi)$ when $\xi = 0$, we have

$$f(0,\eta) = \lim_{s\to\infty} sF(s,\eta), \quad u(0,\eta) = \lim_{s\to\infty} s\,[sF - f(0,\eta)]$$
$$v(0,\eta) = \lim_{s\to\infty} \lambda_f s\Big[\frac{dF}{d\eta} - sF + f(0,\eta)\Big] \qquad (3.8)$$

For values of $\xi \to 0$, the function $\omega_a(\xi)$ can be expanded in a series

$$w_a(\xi) = w_a(0) + w_a'(0)\xi' + \dots \quad \text{or} \quad W_a(s) = w_a(0)s^{-1} + w_a'(0)s^{-2} + \dots \tag{3.9}$$

For high values of s, the functions δ or δ^{-1} are represented in the form of the following series:

$$\delta = [illegible] \dots, \quad \delta^{-1} = 1 - [illegible] + \dots, \quad [illegible] \tag{3.10}$$

Substituting (3.9) and (3.10) in (3.8), and approaching the limit, we obtain

$$f(\xi, \eta) = 0, \qquad u(0, \eta) = -\lambda_f^{-1} w_a(0)\, e^{-\kappa\eta}, \quad v = w_a(0)\, e^{-\kappa\eta} \tag{3.11}$$

Equations (3.11) substantiate the assumption made during the derivation of equation (3.6). It is evident from these equations that velocity disturbances near $\xi = 0$ decrease exponentially as $\eta \to \infty$.

Let us calculate the pressure coefficient on the surface of the airfoil. From (3.5) for the transformed pressure coefficient when $\eta = 0$, we have

$$L[{}^1/_2 C_p] = (i\Gamma + s) W_a(s) [illegible] \tag{3.12}$$

In order to utilize the expression (3.12) let us rewrite it in the form

$$L[C_p/2] = (i\Gamma + s) W_a(s) \lambda_f^{-1} s G(s) C(s)$$

$$G(s) = \frac{[illegible]}{\sqrt{(s+R)^2 + S^2}}, \quad R = a\left(\frac{1}{3} + \frac{3\lambda_f^2 + 2}{3\lambda_e^2} i\Gamma + \frac{\mu + \gamma}{2a}\right), \quad [illegible]$$

[illegible] /41

$$\mu = (3\lambda_f [illegible], \quad \gamma = [illegible] (2N^{1/2})]^{1/2}$$

$$M = 2i\Gamma [illegible] - 2\lambda_e^6$$

$$N = \Gamma [illegible] + \lambda_e^4 \tag{3.13}$$

Applying the convolution theorem twice to the expression (3.13), we arrive at the following formula for the value of the pressure coefficient on the airfoil:

$$\frac{1}{2} C_p = -\frac{1}{\lambda_f}\left\{w_a(0)\Psi(\xi) + \int_0^\xi [w_a' + i\Gamma w_a(\theta)]\Psi(\xi - \theta)\, d\theta\right\}$$

$$\Psi(\xi) = c(\xi) + \int_0^\xi g_0(\theta) c(\xi - \theta)\, d\theta, \quad g(\xi) = L^{-1}[G(s)] = \exp(-R\xi) J_0(S\xi) \tag{3.14}$$

$$c(\xi) = L^{-1}[\dot{C}(s)] = \exp(-m\xi) I_0(n\xi) + (1 + i\Gamma)\int_0^{\xi} \exp(-m\theta) I_0(n\theta)\, d\theta$$

$$m = {}^1/_2(aQ + 1 + i\Gamma), \qquad n = {}^1/_2(aQ - 1 - i\Gamma)$$

Here J_0 and I_0 are the Bessel function and the modified Bessel function of the first kind of zero order, respectively. When $\omega = 0$ (the steady case), $\Psi(\xi) = c(\xi)$, and formula (3.14) coincides with the formula obtained in reference [5] for the case of the steady supersonic nonequilibrium flow around a thin airfoil.

Formula (3.14) can be simplified if we take into consideration that, in the case examined, when in the disturbed flow there takes place only one nonequilibrium process, the values of the stagnation and equilibrium velocities of sound a_f and a_e are close to each other [6]. Thus, for example, in reference [1], a suitable value for a_f/a_e is considered to be 11/10, while in reference [7] a value of 1.16 is chosen for a_f/a_e. Assuming that $\lambda_e^2/\lambda_f^2(1+\varepsilon)$, therefore, where ε is a small quantity (we shall disregard the squares and higher powers of ε), we transform expression (3.12) into the form

$$L\left[\frac{C}{s}\right] = \text{[illegible]} \frac{i\Gamma M_f^2}{\text{[illegible]}} \text{[illegible]} \left\{1 - \text{[illegible]} (M_f + 1 \text{[illegible]})(M_f \text{[illegible]})\right\} \tag{3.15}$$

Returning to expression (3.15), we obtain the following simplified formula for the pressure on the airfoil:

$$\text{[illegible]}(\rho) +$$

$$+ \int_0^{\rho} [g_\rho(\rho - \theta) + i\Gamma g(\rho - \theta)]\, w_a(\theta)\, d\theta \Big\} \Big\{ \frac{\lambda_f^2 \exp[-(1 + i\Gamma)(\xi - \rho)]}{\lambda_f^2 - 2i\Gamma + \Gamma^2} -$$

$$- \frac{\text{[illegible]}(M_f - 1)\exp[-i\Gamma M_f (M_f + 1)^{-1}(\xi - \rho)]}{2iM_f(M_f + 1 + i\Gamma)} + \frac{\Gamma(M_f + 1)\exp[-i\Gamma M_f (M_f - 1)^{-1}(\xi - \rho)]}{2iM_f(M_f - 1 - i\Gamma)} \Big\} d\rho$$

$$g(\xi) = \exp\left(-\frac{i\Gamma M_f^2}{\lambda_f^2}\xi\right) J_0\left(\frac{\Gamma M_f}{\lambda_f^2}\xi\right) \tag{3.16}$$

When $\omega = 0$, we have /42

$$\frac{c}{\text{[illegible]}} = \text{[illegible]}\left[\text{[illegible]}(\xi) - \frac{\varepsilon}{2}\int_0^{\xi} w_a(\theta) \exp[-(\xi - \theta)]\, d\theta\right]$$

which, with an accuracy to ε^2, coincides with the formula obtained in reference

[6] under analogous assumptions, but by another method, for the case of supersonic steady flow over a thin airfoil.

4. For small values of the parameter Γ, i.e., in the case of low-frequency harmonic vibrations of a thin airfoil, when the turbulent flow is almost at equilibrium, the solution of equation (3.6) can be found in the form of an expansion in a series with respect to parameter Γ

$$F = F_0 + \Gamma F_1 + \Gamma^2 F_2 + \ldots \tag{4.1}$$

Substituting (4.1) into (3.6), we obtain the following equations for the zero and first approximations, respectively:

$$\frac{d^2F_0}{d\eta^2} - 2s\frac{dF_0}{d\eta} - \frac{a-1}{1+s}s^2F_0 = 0 \tag{4.2}$$

$$\frac{d^2F_1}{d\eta^2} - 2s\frac{dF_1}{d\eta} - \frac{a-1}{1+s}s^2F_1 = i\frac{M_f^2 s}{\lambda_f(1+s)^2}\frac{[2s^2 + (B^2+3)s + 2B^2]F_0}{\left(B^2 - M_f^2\right)} \tag{4.3}$$

Let us expand the function $W_a(s)$ in a series for Γ.

$$W_a(s) = W_{a0} + \Gamma W_{a1} + \ldots$$

We obtain the boundary conditions for functions $F_0(s,\eta)$ and $F_1(s,\eta)$

$$\lambda_f\left(\frac{dF_0}{d\eta} - sF_0\right) = W_{a0}(s), \qquad \lambda_f\left(\frac{dF_1}{d\eta} - sF_1\right) = W_{a1}(s) \tag{4.4}$$

Equation (4.2), together with the boundary conditions (4.4), describe the solution of the steady state problem examined in reference [5]. For $F_0(s,\eta)$ we have

$$F_0 = -W_{a0}(s)\Omega(s)e^{-s(\delta_0-1)\eta}, \quad \Omega(s) = \frac{1}{\lambda_f s\delta_0}, \quad \delta_0 = \left(\frac{a+s}{1+s}\right)^{1/2} \tag{4.5}$$

The solution of equation (4.3), which satisfies the boundary conditions (4.4) and the condition of attenuation of disturbances at infinity, has the form

$$F_1 = [illegible]\; W_{a0} \; [illegible]\; e^{-s(\delta_0-1)\eta}$$

$$\Theta(s) = \frac{M_f^2[2s^2 + (B^2+3)s + 2B^2]}{2\lambda_f^2 s(1+s)(a+s)} \tag{4.6}$$

For the transformed value of the dimensionless pressure gradient P on the airfoil, which characterizes the local specific lift of the foil, taking into account (3.5) and (4.1) we have

$$L[P] = L\left[\frac{p(\xi,-0) - p(\xi,+0)}{\text{[illegible]}}\right] = 4[sF_0 + \Gamma(sF_1 + iF_0)] \tag{4.7}$$

or substituting expressions (4.5) and (4.6) with $\eta = 0$ into (4.7)

$$L[P] = -4[sW_a(s)\Omega(s) - i\Gamma sW_{a0}(s)\Omega(s)\Theta(s) + i\Gamma W_{a0}(s)\Omega(s)] \tag{4.8}$$

Converting (4.8), we obtain for the pressure gradient on the airfoil: /43

$$P = -\frac{4}{\lambda_f}\left[w_a(0)c(\xi) + \int_0^\xi w_a'(\theta)c(\xi-\theta)\,d\theta - i\Gamma\int_0^\xi G(\theta)E(\xi-\theta)\,d\theta + \right.$$
$$\left. + i\Gamma\int_0^\xi w_{a0}(\theta)c(\xi-\theta)\,d\theta\right]$$
$$c(\xi) = L^{-1}[(s\delta_0)^{-1}] = \exp\left(-\frac{a+1}{2}\xi\right)I_0\left(\frac{a-1}{2}\xi\right) +$$
$$+\int_0^\xi \exp\left(-\frac{a+1}{2}\theta\right)I_0\left(\frac{a-1}{2}\theta\right)d\theta \tag{4.9}$$
$$G(\xi) = L^{-1}[sW_{a0}(s)\Omega(s)] = \frac{1}{\lambda_f}\left[w_{a0}(0)c(\xi) + \int_0^\xi w_{a0}'(\theta)c(\xi-\theta)\,d\theta\right]$$
$$E(\xi) = L^{-1}[\Theta(s)] = \frac{M_e^2}{\lambda_e^2} - \frac{1}{2}e^{-\xi} + \frac{\text{[illegible]} + 2B^2}{2\lambda_f^2 a(a-1)}e^{-a\xi}$$

As an example of the utilization of formula (4.9), let us examine the problem of the instability of longitudinal harmonic oscillations of an airfoil with one degree of freedom. Harmonic longitudinal oscillations with respect to the axis $\xi = \xi_0$ are described by displacement distribution $h(\xi,t) = a(\xi_0-\xi)\exp(i\Gamma t)$.

Hence,

$$w_a(\xi) = -a[1 + i\Gamma(\xi - \xi_0)] = w_{a0} + \Gamma w_{a1} \tag{4.10}$$

Substituting (4.10) into (4.9), we get

$$P(\xi) = \frac{4\alpha}{\lambda_f}[(1 - i\Gamma\xi_0)c(\xi) + i\Gamma A_1(\xi) - i\Gamma A_2(\xi) + i\Gamma A_1(\xi)]$$
$$A_1(\xi) = \int_0^\xi c(\theta)\,d\theta, \qquad A_2(\xi) = \int_0^\xi c(\theta)E(\xi-\theta)\,d\theta \tag{4.11}$$

The first two terms in the brackets describe quasi-stationary flow, while the last two terms represent the compensation for the nonstationary character of the flow. The dimensionless moment (positive, if it acts in the direction of an increase in the angle of attack), which is dependent upon the obtained lift distribution $P(\xi)$, is equal to

$$C_M = \int_0^\Lambda P(\xi)(\xi_0 - \xi)\, d\xi = \alpha\left[C_{M_\alpha} + i\Gamma\left(C_{M_q} + C_{M_{\dot\alpha}}\right)\right] \tag{4.12}$$

where: $\Lambda = 1/1_r$ = the ratio of the length of the airfoil to the relaxation length; C_{M_α} = the static moment; C_{M_q} = the quasi-stationary longitudinal damping; and $C_{M_{\dot\alpha}}$ = a compensation related to the instability of the flow.

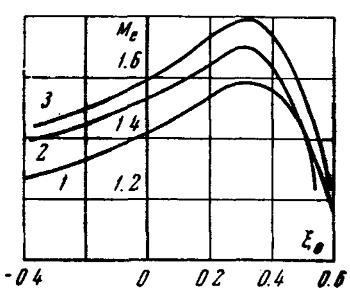
Inserting (4.11) into (4.12), we find

$$C_{M_\alpha} = \frac{4}{\lambda_f}\left[\xi_0 A_1(\Lambda) - A_3(\Lambda)\right], \quad A_3(\xi) = \int_0^\xi \theta c(\theta)\, d\theta \tag{4.13}$$

$$C_{M_q} = -\frac{4}{\lambda_f}\left[\xi_0^2 A_1(\Lambda) - \xi_0 A_3(\Lambda) + \int_0^\Lambda (\xi - \xi_0) A_1(\xi)\, d\xi\right] \tag{4.14}$$

$$C_{M_{\dot\alpha}} = \frac{4}{\lambda_f}\left[\int_0^\Lambda (\xi - \xi_0) A_2(\xi)\, d\xi - \int_0^\Lambda (\xi - \xi_0) A_1(\xi)\, d\xi\right] \tag{4.15}$$

For some values of the number M_f and of the abscissa ξ_0, the value of /44
the total moment $C_{M_q} + C_{M_{\dot\alpha}}$, which characterizes damping, may be negative. Thus, the stability boundary of longitudinal harmonic oscillations of an airfoil with one degree of freedom can be found, if we assume that



$$C_{M_q} + C_{M_{\dot\alpha}} = 0 \tag{4.16}$$

The figure illustrates curves (stability boundaries) obtained from equation (4.16) by numerical integration of the expressions (4.15) and (4.14) for $\Lambda = \infty$ (the equilibrium case) and for $\Lambda = 1$ and 0.4 (curves 1,2, and 3, respectively). As is evident from an examination of the curves, the presence of a nonequilibrium process increases the region of instability of oscillations (the region inside the curves).

5. With larger values of Γ, i.e., in the cases of high-frequency harmonic oscillations of a thin airfoil in a supersonic nonequilibrium stream of gas, when the turbulent flow is close to stagnation, equation (3.2) can be solved by the method of asymptotic expansion of the turbulent velocity potential with respect to parameter Γ [8]. In this method the boundary condition (3.3) is satisfied on the surface of the airfoil, and not when $z = 0$, i.e., we have the bound-

ary condition

$$\partial\varphi/\partial n = \omega \quad \text{when } n = 0 \tag{5.1}$$

(where: n = dimensionless normal to the surface of the airfoil; ω = dimensionless component of velocity normal to the surface of the airfoil), which allows us to take into account the shape of the airfoil in the final formulas. The turbulent velocity potential must also satisfy the radiation condition, in accordance with which the function $\varphi(x,z)\exp(i\Gamma t)$ must describe a divergent wave of finite intensity at infinity.

Let us represent the function $\varphi(x,z)$ in the form

$$\varphi(x, z) = \Gamma^m [\Gamma^{-1}\chi_1(x, z) + \Gamma^{-2}\chi_2(x, z) + \ldots] \exp[i\Gamma\psi(x, z)] \tag{5.2}$$

Substituting (5.2) into (3.2) and (5.1), and equating the coefficients of Γ of equal degrees to zero, we get the following equations for determining the functions $\Psi(x,z)$ and $\chi_1(x,z)$:

$$\lambda_f^2\psi_x^3 - (3M_f^2 - 1)\psi_x^2 + \psi_z^2 - \psi_z^2\psi_x + 3M_f^2\psi_x - M_f^2 = 0 \tag{5.3}$$

$$\begin{aligned} &[\psi_z^2 - 3\lambda_f^2\psi_x^2 + 2(3M_f^2 - 1)\psi_x - 3M_f^2]\chi_{1x} + 2\psi_z(\psi_x - 1)\chi_{1z} + \\ &+ [(3M_f^2 - 1)\psi_{xx} + \psi_{zz}(\psi_x - 1) - 3\lambda_f^2\psi_x\psi_{xx} + 2\psi_z\psi_{xz} - \\ &- \lambda_e^2\psi_x^2 + \psi_z^2 + 2M_e^2\psi_x - M_e^2]\chi_1 = 0 \end{aligned} \tag{5.4}$$

(here and subsequently the indices represent differentiation with respect to the corresponding coordinates). The equations for the determination of the function $\chi_r(x,z)$, $(r \geqslant 2)$ are analogous to equation (5.4) with the right-hand side known. The boundary conditions for the functions $\psi(x,z)$ and $\chi_r(x,z)$ will be

$$\psi(x, z) = 0, \quad \chi_1\frac{\partial\psi}{\partial n} = iw, \quad \chi_r\frac{\partial\psi}{\partial n} = -i\frac{\partial\chi_{r-1}}{\partial n} \quad (r \geqslant 2) \quad \text{when } n = 0 \tag{5.5}$$

From the condition (5.1) it also follows that m = 0.

The solution of equation (5.3) is equivalent to the solution of the characteristic system of equations: /45

$$\begin{gathered} \frac{dx}{d\sigma} = H_p, \quad \frac{dz}{d\sigma} = H_q, \quad \frac{d\psi}{d\sigma} = pH_p + qH_q, \quad p = \psi_x \\ \frac{dp}{d\sigma} = -(pH_\psi + H_x), \quad \frac{dq}{d\sigma} = -(qH_\psi + H_z) \quad q = \psi_z \\ 2H = [(M_f^2 - 1)p^3 - (3M_f^2 - 1)p^2 + q^2(1 - p) + 3M_f^2p - M_f^2], \end{gathered} \tag{5.6}$$

The parameter σ varies along the characteristic lines of the solution.

The differential equation (5.3) will be $H = 0$.

The boundary conditions for the characteristic system (5.6) will be (we shall **assume that** on the surface of the airfoil $\sigma = 0$)

$$\psi = 0, \; x = x_0(\tau), \; z = z_0(\tau), \; p = p_0(\tau), \; q = q_0(\tau) \quad \text{when } \sigma = 0 \tag{5.7}$$

where $x = x_0(\tau)$ and $z = z_0(\tau)$ represent dimensionless parametric equations of the surface of the airfoil (the parameter τ increases as it travels over the surface of the airfoil in a clockwise direction; the motion of the airfoil in parametric form is given in the form $X = x_0(\tau)\exp(i\Gamma t)$, $Z = z_0(\tau)\exp(i\Gamma t)$.

The quantities $p_0(\tau)$ and $q_0(\tau)$ can be determined from the equation

$$H\,[p_0(\tau), q_0(\tau)] = 0$$

and the constancy conditions $\Psi(x,z)$ on the surface of the airfoil

$$p_0(\tau)x_0'(\tau) + q_0(\tau)z_0'(\tau) = 0$$

and for the flow which satisfies the radiation condition (i.e., describes divergent waves) are equal to

$$p = -\frac{M_f \sin\theta(\tau)}{1 - M_f \sin\theta(\tau)}, \quad q_0 = \frac{M_f \cos\theta(\tau)}{1 - M_f \sin\theta(\tau)}, \quad \begin{array}{l} x_0'(\tau) = l'(\tau)\cos\theta(\tau) \\ z_0'(\tau) = l'(\tau)\sin\theta(\tau) \end{array} \tag{5.8}$$

Here $\theta(\tau)$ is the slope of the airfoil contour; $l(\tau)$ is the dimensionless length along the airfoil contour. Solving the characteristic system (5.6) with the boundary conditions (5.7) and (5.8), we get

$$p = p_0, \; q = q_0, \; x = x_0(\tau) + \frac{M_f - \sin\theta(\tau)}{1 - M_f \sin\theta(\tau)}\sigma, \quad \psi = \frac{M_f}{1 - M_f \sin\theta(\tau)}\sigma$$
$$z = z_0(\tau) + \frac{\cos\theta(\tau)}{1 - M_f \sin\theta(\tau)}\sigma \tag{5.9}$$

or, **eliminating** σ,

$$z = z_0(\tau) + \frac{\cos\theta(\tau)}{M_f}\psi, \quad x = x_0(\tau) + \frac{M_f - \sin\theta(\tau)}{M_f}\psi$$

Thus, the solution for function $\Psi(x,z)$ coincides with the corresponding solution for Ψ in the case of an inert gas [2], where, however, the Mach number is replaced by the stagnation Mach number M_f.

To determine function $\chi_1(x,z)$ let us cross from the coordinates x,z to the characteristic coordinates τ,σ with the help of the transition formulas

$$\frac{\partial\tau}{\partial x}=\frac{\Delta\cos\theta(\tau)}{l'(\tau)},\qquad \frac{\partial\tau}{\partial z}=-\frac{\Delta[M_f-\sin\theta(\tau)]}{l'(\tau)}$$

$$\frac{\partial\sigma}{\partial x}=\Delta\left[-\sin\theta(1-M_f\sin\theta)+\frac{M_f-\sin\theta}{1-M_f\sin\theta}\frac{\sigma}{R}\right]$$

$$\frac{\partial\sigma}{\partial z}=\Delta\cos\theta\left[1-M_f\sin\theta-\frac{M_f^2-1}{1-M_f\sin\theta}\frac{\sigma}{R}\right]\quad \Delta=[1-M_f\sin\theta(\tau)+R^{-1}(\tau)\sigma]^{-1} \tag{5.10}$$

Here $R(\tau) = -l'(\tau)/\theta'(\tau)$ is the dimensionless radius of curvature of the contour. With the new coordinates, equation (5.4) takes on the form /46

$$\frac{\partial\chi_1}{\partial\sigma}+\frac{1}{2}\left\{\frac{1}{[1-M_f\sin\theta(\tau)]R(\tau)+\sigma}+\frac{M_f(B^2-1)}{[1-M_f\sin\theta(\tau)]}\right\}\chi_1=0 \tag{5.11}$$

while for the boundary conditions, when $\sigma = 0$, we have from (5.5)

$$\chi_2\frac{\partial\psi}{\partial n}=\chi_2\left(-\frac{\partial\psi}{\partial x}\sin\theta+\frac{\partial\psi}{\partial z}\cos\theta\right)=\frac{M_f}{1-M_f\sin\theta(\tau)}\chi_1=iw(\tau) \tag{5.12}$$

Solving equation (5.11) with boundary condition (5.12), for the turbulent velocity potential from (5.2) and (5.9) we obtain

$$\varphi=\frac{iw\Gamma^{-1}}{M_f}(1-M_f\sin\theta)\left[1+\frac{\sigma}{(1-M_f\sin\theta)R}\right]^{-1/2}\times$$
$$\times\exp\left[-\left(i\Gamma+\frac{B^2-1}{2}\right)M_f(1-M_f\sin\theta)^{-1}\sigma\right]. \tag{5.13}$$

It is readily seen that the turbulent velocity potential satisfies the radiation condition as well.

Let us determine the pressure coefficient on the surface of the airfoil. From (3.3) and (5.2) we have (when $\sigma = 0$)

$$\frac{C_p}{2}=-2\left(i\Gamma\varphi+\frac{\partial\varphi}{\partial x}\right)=-2\left[i\frac{\text{[illegible]}}{1-M_f\sin\theta}+\Gamma^{-1}\left(\frac{i\chi_2}{1-M_f\sin\theta}+\chi_{1x}\right)\right] \tag{5.14}$$

From the boundary conditions (5.5) we have

$$\chi_2\frac{\partial\psi}{\partial n}=-i\frac{\partial\chi_1}{\partial n},\quad \chi_2=-\frac{i(1-M_f\sin\theta)}{M_f}(\chi_{1z}\cos\theta-\chi_{1x}\sin\theta)\ \text{when}\ \sigma=0 \tag{5.15}$$

Substituting (5.15) into (5.14) and transforming into the coordinates τ and σ, we get

$$C_p = -2\left[\frac{i\chi_1}{1 - M_f \sin\theta} + \Gamma^{-1}\frac{1 - M_f \sin\theta}{M_f}\frac{\partial\chi_1}{\partial\sigma} + O(\Gamma^{-2})\right] \quad (5.16)$$

The values of the functions χ_1 and $\partial\chi_1/\partial\sigma$ when $\sigma = 0$ can be derived from (5.11) and (5.12). Then, for the amplitude of the pressure coefficient on the surface of the section, taking into consideration the first two approximations in the expansion (5.2), we obtain

$$C_p = 2\frac{w}{M_f}\left\{1 + i\Gamma^{-1}\frac{1 - M_f \sin\theta}{2M_f}\left[\frac{1}{K} + M_f(B^2 - 1)\right]\right\} \quad (5.17)$$

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